A sharp lower bound on the number of non-equivalent colorings of graphs of order n and maximum degree n − 3

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A sharp lower bound on the number of non-equivalent colorings of graphs of order n and maximum degree $n-3$

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Abstract.
Two vertex colorings of a graph G are equivalent if they induce the same partition of the vertex set into color classes. The graphical Bell number $\mathcal{B}(G)$ is the number of non-equivalent vertex colorings of G. We determine a sharp lower bound on $\mathcal{B}(G)$ for graphs G of order n and maximum degree $n-3$, and we characterize the graphs for which the bound is attained.

Keywords: non-equivalent colorings; graphical Bell number; extremal theory; fixed maximum degree.

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1 Introduction

A coloring of a graph G is the assignment of a color to each vertex of G such that no two adjacent vertices share the same color. A subset of vertices assigned to the same color is called a color class. Two colorings of G are said *equivalent* if they induce the same partition of the vertex set into color classes. The *graphical Bell number* $\mathcal{B}(G)$ is the number of non-equivalent colorings of G. This invariant has been studied by

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several authors in the last few years [6–8, 10]. It is related to the σ-polynomial of a graph G [2, 3, 11] defined as the polynomial in x such that the coefficient of x^k is the number of non-equivalent colorings of G with exactly k non-empty color classes. Hence, $\mathcal{B}(G)$ is the value of the σ-polynomial at $x = 1$. $\mathcal{B}(G)$ is also related to the standard *Bell number* B_n (sequence A000110 in OEIS [13]) that corresponds to the number of partitions of a set of n elements into non-empty subsets, and is thus obviously the same as the number of non-equivalent colorings of a graph of order n and without any edge.

Extremal properties of $\mathcal{B}(G)$ were recently studied in [9]. In particular, the authors give a sharp upper bound on $\mathcal{B}(G)$ for graphs of order n and any fixed maximum degree. Also, given two positive integers n and r with $0 \leq r \leq n-1$, let $\mathscr{G}_r^n$ denote the class of graphs of order n and maximum degree r. A graph $G \in \mathscr{G}_r^n$ is said *extremal* if $\mathcal{B}(G) = \min_{H \in \mathscr{G}_r^n} \mathcal{B}(H)$. Extremal graphs in $\mathscr{G}_r^n$ are characterized in [9] for $r = 1, 2, n-2$ and $n-1$. We continue along the same path by considering the case $r = n-3$. In other words, we characterize the graphs with maximum degree $n-3$ that minimize $\mathcal{B}(G)$, and thus determine a sharp lower bound on the number of non-equivalent colorings $\mathcal{B}(G)$ for graphs G in $\mathscr{G}_{n-3}^n$. The results presented here were first conjectured with the help of the system GraPHedron [12] and later outputted again by Digenes [1]. Both of these tools are conjecture-making systems helping discovery in graph theory.

The paper is organized as follows. In the next section, we fix some notations, give additional definitions, and recall some basic properties of $\mathcal{B}(G)$. In Section 3, we give a new proof of the sharp lower bound on $\mathcal{B}(G)$ for graphs G in $\mathscr{G}_{n-2}^n$. It is based on similar arguments as those that we will use for proving our main theorem. The sharp lower bound on $\mathcal{B}(G)$ for graphs G of order n and maximum degree $n-3$ is established in Section 4, together with a characterization of the extremal graphs in $\mathscr{G}_{n-3}^n$.

2 Notations and basic properties

We refer to the book of Diestel [4] for basic notations and definitions in graph theory. Let $G = (V, E)$ be a simple undirected graph, we denote by $n = |V|$ and $m = |E|$ the order and size of G, respectively. We write $G \simeq H$ if two graphs G and H are isomorphic. For a subset $W \subseteq V$ of vertices, we denote by $G[W]$ the subgraph of G induced by W, while $G \setminus W$ denotes the subgraph of G induced by $V \setminus W$ (i.e., $G \setminus W = G[V \setminus W]$). The *disjoint union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \emptyset$ is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$, while the *join* of G_1 and G_2 is the graph $G_1 + G_2 = (V_1 \cup V_2, E)$ such that E contains all edges in $E_1 \cup E_2$ as well as all edges linking a vertex in V_1 with a vertex in V_2. We denote by pG the disjoint union of p copies of G.

Let u and v be two vertices of a graph $G = (V, E)$, we denote by $G_{|uv}$ the graph obtained by identifying (merging) vertices u and v, and removing the edge linking them, if any. If u and v are two adjacent vertices in G, we denote by $G - uv$ the graph obtained by removing the edge that links u with v in G. Similarly, if u and v are two non-adjacent vertices in G, we denote by $G + uv$ the graph obtained by adding an edge between u and v.

We write $\overline{G}$ for the *graph complement* of G, while $\mathsf{K_n}$ denotes the *complete graph* of order n, $\mathsf{P_n}$ the *path* of order n, and $\mathsf{C_n}$ the *cycle* of order n. Also, we write $\mathsf{K}_{a,b}$ for the *complete bipartite graph* where a and b are the cardinalities of the two sets of vertices of the bipartition, and $\mathsf{S_n}$ denotes the *star* on n vertices, that is $\mathsf{K}_{1,n-1}$.

The *degree* $d(v)$ of a vertex v in $G = (V, E)$ is the number of vertices adjacent to v. Vertex v is *isolated* if $d(v) = 0$, it is *dominating* (or *universal*) if $d(v) = |V| - 1$, and $\Delta(G)$ denotes the maximum degree of G.

A *clique* in a graph G is a set of pairwise adjacent vertices, while a *stable set* is a set of pairwise non-adjacent vertices. A *coloring* of G is the assignment of a color to each vertex of G such that no two adjacent vertices share the same color. A *color class* is a set of vertices with the same color. Each color class is therefore a stable set, and every coloring of G induces a partition of its vertex set into color classes. Two colorings of G are *equivalent* if they induce the same partition of the vertex set into color classes. The graphical Bell number $\mathcal{B}(G)$ of G is the total number of non-equivalent colorings of G. As mentioned in Section 1, given two positive integers n and r with $0 \le r \le n-1$, $\mathscr{G}_r^n$ denotes the class of graphs of order n with $\Delta(G) = r$, and a graph G in $\mathscr{G}_r^n$ is *extremal* if $\mathcal{B}(G) \le \mathcal{B}(H)$ for all $H \in \mathscr{G}_r^n$.

The following *deletion-contraction* rule (also called the *Fundamental Reduction Theorem* [5]) is a well known method to compute $\mathcal{B}(G)$ [7, 10].

Property 1. [7, 10] *Let u and v be two vertices of a graph G:*

$$\mathcal{B}(G) = \mathcal{B}(G - uv) - \mathcal{B}(G_{|uv}) \quad \text{if } u \text{ and } v \text{ are adjacent vertices} \tag{1}$$

$$\mathcal{B}(G) = \mathcal{B}(G + uv) + \mathcal{B}(G_{|uv}) \quad \text{if } u \text{ and } v \text{ are non-adjacent vertices.} \tag{2}$$

The following property is a direct consequence of the fact that a dominating vertex in a graph G is alone in its color class for all colorings of G.

Property 2. [7, 9] *If v is a dominating vertex of G, then $\mathcal{B}(G) = \mathcal{B}(G \setminus \{v\})$.*

The next property is very useful for recognizing extremal graphs.

Property 3. [9] *If u and v are two non-adjacent vertices in an extremal graph G, then $\max\{d(u), d(v)\} = \Delta(G)$.*

The following simple property shows how to compute $\mathcal{B}(G)$ for a join $G = G_1 + G_2$. It follows from the fact that none of the vertices of G_1 can share a color with a vertex of G_2.

Property 4. *If $G = G_1 + G_2$ then $\mathcal{B}(G) = \mathcal{B}(G_1)\mathcal{B}(G_2)$.*

Since the complement of a join is the disjoint union of two graphs, we have the following corollary.

Corollary 5. *Let G be a graph, and let $X_1, X_2, \ldots, X_k$ denote the vertex sets of the connected components of $\overline{G}$. Then*

$$\mathcal{B}(G) = \prod_{i=1}^{k} \mathcal{B}(G[X_i]).$$

The next two properties link the graphical Bell number with Fibonacci and Lucas numbers F_n and L_n (sequences A000045 and A000204 in OEIS [13]). They also give the values of the graphical Bell number of complements of paths and cycles, which will be very useful for the proofs contained in the next sections.

Property 6. [7,10] $\mathcal{B}(\overline{\mathsf{P_n}}) = \mathsf{F}_{n+1}$ *for all* $n \geq 1$.

Property 7. [7,10] $\mathcal{B}(\overline{\mathsf{C}_n}) = \mathsf{L}_n$ *for all* $n \geq 4$.

3 Extremal graphs in $\mathscr{G}^n_{n-2}$

The following theorem characterizes the extremal graphs in $\mathscr{G}^n_{n-2}$, and thus gives a sharp lower bound on $\mathcal{B}(G)$ for graphs of order n and maximum degree $n-2$. It was proved in [9], but with a different scheme than that proposed here below. The new proof is based on arguments similar to those we will use in the next section for proving Theorem 14 that characterizes extremal graphs in $\mathscr{G}^n_{n-3}$.

Theorem 8. [9] *Let* $G = (V, E)$ *be a graph of order* $n \geq 2$ *and maximum degree* $\Delta(G) \leq n-2$. *Then*

$$\mathcal{B}(G) \geq n$$

with equality if and only if $G \simeq \mathsf{K}_1 \cup \mathsf{K}_{n-1}$ *when* $n \neq 4$, *and* $G \simeq \mathsf{K}_1 \cup \mathsf{K}_3$ *or* $G \simeq \mathsf{C}_4$ *otherwise.*

Proof. We can assume $G \in \mathscr{G}^n_{n-2}$ since $\mathcal{B}(G) > \mathcal{B}(G+uv)$ for every pair of non-adjacent vertices u, v in G. We proceed by induction on n. The result is clearly valid for $n = 2$ since G then contains two isolated vertices, which means that $G \simeq \mathsf{K}_1 \cup \mathsf{K}_1$, and $\mathcal{B}(G) = 2$. So assume $n > 2$ and suppose G is extremal in $\mathscr{G}^n_{n-2}$. Notice first that $\mathcal{B}(\mathsf{K}_1 \cup \mathsf{K}_{n-1}) = n$ because either the isolated vertex of K_1 has its own color, or it uses one of the $n-1$ colors in K_{n-1}. Hence, $\mathcal{B}(G) \leq n$. Since $\mathcal{B}(\mathsf{C}_4) = 4$, it remains to prove that $G \simeq \mathsf{K}_1 \cup \mathsf{K}_{n-1}$ when $n \neq 4$, and $G \simeq \mathsf{K}_3 \cup \mathsf{K}_1$ or $G \simeq \mathsf{C}_4$ otherwise. Equivalently, we have to prove that $\overline{G} \simeq \mathsf{S_n}$ when $n \neq 4$, and $\overline{G} \simeq \mathsf{S}_4$ or $2\mathsf{K}_2$ otherwise.

Let A be the set of vertices of degree 1 in $\overline{G}$ and let $B = V \setminus A$. Then $\overline{G}[A]$ is the disjoint union of edges and isolated vertices, that is $\overline{G}[A] = p\mathsf{K}_2 \cup q\mathsf{K}_1$ with $2p+q = |A|$. Moreover, every edge in $\overline{G}[A]$ is a connected component of $\overline{G}$. Also, since G is extremal, we know from Property 3 that $\overline{G}[B]$ is a stable set, which means that in $\overline{G}$, every vertex in B has at least two neighbors in A.

- If $p \geq 1$, then consider two adjacent vertices u and v in $\overline{G}[A]$. Note that $\overline{G}[\{u,v\}]$ is a connected component of $\overline{G}$ and $\mathcal{B}(G[\{u,v\}]) = 2$. Moreover, $n \geq 4$ because $G \setminus \{u,v\}$ contains at least one vertex in A and another one in $A \cup B$. Since $G \setminus \{u,v\}$ has $n-2$ vertices and maximum degree $n-4$, we know by induction that $\mathcal{B}(G \setminus \{u,v\}) \geq n-2$. It then follows from Corollary 5 that

  $$n \geq \mathcal{B}(G) = \mathcal{B}(G[\{u,v\}])\mathcal{B}(G \setminus \{u,v\}) \geq 2(n-2).$$

  We therefore have $n = 4$, which means that $\overline{G} \simeq 2\mathsf{K}_2$.

- If $p = 0$ then $\overline{G}[A]$ is a stable set, which means that $\overline{G}$ is the disjoint union of stars $\mathsf{S}_{n_1}, \ldots, \mathsf{S}_{n_r}$ with $\sum_{i=1}^{r} n_i = n$ and $n_i \geq 3$ $(i = 1, \ldots, r)$. If $r = 1$ then $\overline{G} \simeq \mathsf{S_n}$. So assume $r > 1$ and let W be the vertex set of S_{n_1}. It then follows from Corollary 5 that $\mathcal{B}(G) = \mathcal{B}(G[W])\mathcal{B}(G \setminus W)$. Since $G[W]$ has n_1 vertices and maximum degree $n_1 - 2$ while $G \setminus W$ has $n - n_1$ vertices and maximum degree $(n - n_1) - 2$, we get a contradiction, using the induction hypothesis:

$$\begin{aligned} n \geq \mathcal{B}(G) \geq n_1(n - n_1) &= n + (n_1 - 1)n - n_1^2 \\ &\geq n + (n_1 - 1)(n_1 + 3) - n_1^2 = n + 2n_1 - 3 \\ &> n. \end{aligned}$$ □

4 Extremal graphs in $\mathscr{G}_{n-3}^{n}$

We begin this section with some simple observations about graphs in $\mathscr{G}_{n-3}^{n}$:

- if $n = 3$, then $\overline{\mathsf{C}_3}$ is the only graph in $\mathscr{G}_{n-3}^{n}$;
- if $n \geq 4$, then both $\overline{\mathsf{C}_n}$ and $\mathsf{K}_2 \cup \mathsf{K}_{n-2}$ belong to $\mathscr{G}_{n-3}^{n}$.

Hence, for an extremal graph $G \in \mathscr{G}_{n-3}^{n}$, we have $\mathcal{B}(G) = \mathcal{B}(\overline{\mathsf{C}_3}) = 5$ if $n = 3$, and $\mathcal{B}(G) \leq \min\{\mathcal{B}(\overline{\mathsf{C}_n}), \mathcal{B}(\mathsf{K}_2 \cup \mathsf{K}_{n-2})\}$ if $n \geq 4$. We know from Property 7 that $\mathcal{B}(\overline{\mathsf{C}_n}) = \mathsf{L}_n$ for $n \geq 4$. The next lemma gives the value of $\mathcal{B}(\mathsf{K}_2 \cup \mathsf{K}_{n-2})$ for $n \geq 4$.

Lemma 9. *If $n \geq 4$, then $\mathcal{B}(\mathsf{K}_2 \cup \mathsf{K}_{n-2}) = n^2 - 3n + 3$.*

Proof. Let u and v be the two vertices of the K_2. If vertex u gets the same color as one vertex in K_{n-2}, then v can get any of the $n-3$ other colors used in K_{n-2}, or a new color, which gives a total of $(n-2)^2$ non-equivalent such colorings. If vertex u does not share a color with a vertex in K_{n-2}, then v can have any of the colors used in K_{n-2}, or a new color different from that used by u, which gives a total of $n - 1$ non-equivalent colorings. Hence, $\mathcal{B}(\mathsf{K}_2 \cup \mathsf{K}_{n-2}) = (n-2)^2 + n - 1 = n^2 - 3n + 3$. □

Since $\mathsf{L}_n > n^2 - 3n + 3$ for $n \geq 8$ while $\mathsf{L}_n \leq n^2 - 3n + 3$ for $4 \leq n \leq 7$, we have shown that if G is an extremal graph in $\mathscr{G}_{n-3}^{n}$, then $\mathcal{B}(G) \leq h(n)$ for all $n \geq 3$, where $h(n)$ is defined as follows:

$$h(n) = \begin{cases} 5 & \text{if } n = 3 \\ \mathsf{L}_n & \text{if } 4 \leq n \leq 7 \\ n^2 - 3n + 3 & \text{if } n \geq 8. \end{cases}$$

We will show in Theorem 14 that $\mathcal{B}(G) = h(n)$ for all extremal graphs $G \in \mathscr{G}_{n-3}^{n}$. We first prove some properties of function $h(n)$.

Lemma 10. *Let n and r be two integers such that $n \geq r + 3 \geq 5$. Then*

$$h(n) < \mathsf{F}_{r+1} h(n - r) + 2\mathsf{F}_r (n - r - 1).$$

Proof. Consider any fixed $r \geq 2$, and let $f(n) = \mathsf{F}_{r+1}h(n-r) + 2\mathsf{F}_r(n-r-1)$. We distinguish the following cases.

- If $n - r \geq 8$, then consider function $g(n) = f(n) - h(n)$. It follows from the definition of $h(n)$ that

$$g(n) = \mathsf{F}_{r+1}((n-r)^2 - 3(n-r) + 3) + 2\mathsf{F}_r(n-r-1) - n^2 + 3n - 3.$$

  The first derivative of g is

$$n(2\mathsf{F}_{r+1} - 2) + 2\mathsf{F}_r - 2r\mathsf{F}_{r+1} - 3\mathsf{F}_{r+1} + 3.$$

  It is equal to zero when $n = \dfrac{(2r+3)\mathsf{F}_{r+1} - 2\mathsf{F}_r - 3}{2\mathsf{F}_{r+1} - 2} < r + 3$, and strictly positive for larger values of n. Hence, for $n \geq r + 8$, $g(n)$ is an increasing function, and since $g(r+8) = 43\mathsf{F}_{r+1} + 14\mathsf{F}_r - r^2 - 13r - 43 > 0$ for all $r > 1$, we conclude that $g(n)$ is strictly positive (*i.e.*, $h(n) < f(n)$) for all $n \geq r + 8$.
- If $n - r = 7$ then $f(n) = 29\mathsf{F}_{n-6} + 12\mathsf{F}_{n-7} > n^2 - 3n + 3 = h(n)$ for all $n \geq 9$.
- If $n - r = 6$ then $f(n) = 18\mathsf{F}_{n-5} + 10\mathsf{F}_{n-6} > n^2 - 3n + 3 = h(n)$ for all $n \geq 8$.
- If $n - r = 5$ then $f(n) = 11\mathsf{F}_{n-4} + 8\mathsf{F}_{n-5} > n^2 - 3n + 3 = h(n)$ for all $n \geq 8$ and $f(7) = 30 > 29 = h(7)$.
- If $n - r = 4$ then $f(n) = 7\mathsf{F}_{n-3} + 6\mathsf{F}_{n-4} > n^2 - 3n + 3 \geq h(n)$ for all $n \geq 7$ and $f(6) = 20 > 18 = h(6)$.
- If $n - r = 3$ then $f(n) = 5\mathsf{F}_{n-2} + 4\mathsf{F}_{n-3} > n^2 - 3n + 3 \geq h(n)$ for all $n \geq 5$. □

Lemma 11. *Let n_1, n_2 be two integers such that $n_1 \geq 3$ and $n_2 \geq 3$. Then*

$$h(n_1)h(n_2) > h(n_1 + n_2).$$

Proof. We analyze three cases.

- If both n_1 and n_2 belong to $\{3, 4, 5, 6, 7\}$ then the result can be checked with the values in Table 1.

n	3	4	5	6	7	8	9	10	11	12	13	14
$h(n)$	5	7	11	18	29	43	57	73	91	111	133	157

Table 1: Values of $h(n)$ for $3 \leq n \leq 14$.

- If $n_1 \in \{3, 4, 5, 6, 7\}$ and $n_2 \geq 8$, then the following inequalities (which are easy to verify) prove the result:

$$\begin{array}{rcl}
5(n_2^2 - 3n_2 + 3) & > & (n_2+3)^2 - 3(n_2+3) + 3, \\
7(n_2^2 - 3n_2 + 3) & > & (n_2+4)^2 - 3(n_2+4) + 3, \\
11(n_2^2 - 3n_2 + 3) & > & (n_2+5)^2 - 3(n_2+5) + 3, \\
18(n_2^2 - 3n_2 + 3) & > & (n_2+6)^2 - 3(n_2+6) + 3, \\
29(n_2^2 - 3n_2 + 3) & > & (n_2+7)^2 - 3(n_2+7) + 3.
\end{array}$$

- If both n_1 and n_2 are at least equal to 8, then let $p(n) = n^2 - 3n + 3$ and assume without loss of generality that $n_2 \geq n_1$. We then have

$$\begin{aligned} & h(n_1)h(n_2) - h(n_1 + n_2) = p(n_1)p(n_2) - p(n_1 + n_2) \\ =& (n_1^2 - 3n_1 + 3)p(n_2) - ((n_1 + n_2)^2 - 3(n_1 + n_2) + 3) \\ =& n_1[n_1(p(n_2) - 1) - (3p(n_2) + 2n_2 - 3)] + (3p(n_2) - n_2^2) + (3n_2 - 3). \end{aligned}$$

Observe that the last two terms are strictly positive since $3p(n_2) - n_2^2 > 0$ for all $n_2 \geq 4$ and $3n_2 - 3 > 0$ for all $n_2 \geq 2$. Hence,

$$\begin{aligned} h(n_1)h(n_2) - h(n_1 + n_2) &> n_1(p(n_2) - 1) - (3p(n_2) + 2n_2 - 3) \\ &= (n_1 - 3)p(n_2) - n_1 - 2n_2 + 3 \\ &\geq 5p(n_2) - 3n_2 + 3 = 5n_2^2 - 18n_2 + 18. \end{aligned}$$

Since $5n_2^2 - 18n_2 + 18 > 0$ for $n_2 \geq 0$, we have $h(n_1)h(n_2) - h(n_1 + n_2) > 0$. □

Notation. *Let v be a vertex in a graph G. For an integer $r \geq 1$, we denote by G_v^r the graph obtained by adding an edge between v and one of the endpoint of P_r in $G \cup \mathsf{P}_r$ (see Figure 1). For r = 0, we set $G_v^0 = G$.*

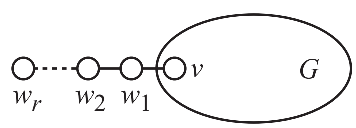

Figure 1: The graph G_v^r.

Lemma 12. *Let v be a vertex in a graph G and consider any integer $r \geq 0$. Then*

$$\mathcal{B}(\overline{G_v^r}) = \mathsf{F}_{r+1}\mathcal{B}(\overline{G}) + \mathsf{F}_r\mathcal{B}(\overline{G} \setminus \{v\}).$$

Proof. If $r = 0$ the result is clearly valid since $\overline{G_v^0} = \overline{G}$, $\mathsf{F}_1 = 1$ and $\mathsf{F}_0 = 0$. So assume $r \geq 1$. Let $\{w_1, \ldots, w_r\}$ be the vertex set of P_r and let $\{w_1w_2, \ldots, w_{r-1}w_r\}$ be its edge set so that G_v^r is obtained from G by adding the edge vw_1 to $G \cup \mathsf{P}_r$. It follows from Equation (2) in Property 1 that $\mathcal{B}(\overline{G_v^r}) = \mathcal{B}(\overline{G_v^r} + w_rw_{r-1}) + \mathcal{B}(\overline{G_v^r}_{|w_rw_{r-1}})$, where $w_0 = v$. Observe that $\overline{G_v^r} + w_rw_{r-1}$ is isomorphic to the graph obtained by adding a dominating vertex to $\overline{G_v^r} \setminus \{w_r\}$ while $\overline{G_v^r}_{|w_rw_{r-1}}$ is isomorphic to the graph obtained by adding a dominating vertex to $\overline{G_v^r} \setminus \{w_r, w_{r-1}\}$. By Property 2, we therefore have

$$\mathcal{B}(\overline{G_v^r}) = \mathcal{B}(\overline{G_v^r} \setminus \{w_r\}) + \mathcal{B}(\overline{G_v^r} \setminus \{w_r, w_{r-1}\}).$$

If $r = 1$ then $\overline{G_v^r} \setminus \{w_r\} = \overline{G}$ and $\overline{G_v^r} \setminus \{w_r, w_{r-1}\} = \overline{G} \setminus \{v\}$, which means that $\mathcal{B}(\overline{G_v^r}) = \mathsf{F}_{r+1}\mathcal{B}(\overline{G}) + \mathsf{F}_r\mathcal{B}(\overline{G} \setminus \{v\})$ (since $\mathsf{F}_2 = \mathsf{F}_1 = 1$). So assume $r > 1$ and suppose the result is valid for smaller values of r. Observe that $\overline{G_v^r}\setminus\{w_r\} \simeq \overline{G_v^{r-1}}$ and $\overline{G_v^r}\setminus\{w_r, w_{r-1}\} \simeq \overline{G_v^{r-2}}$. It then follows from the induction hypothesis that

$$\begin{aligned}\mathcal{B}(\overline{G_v^r}) &= \mathcal{B}(\overline{G_v^{r-1}}) + \mathcal{B}(\overline{G_v^{r-2}})\\ &= \mathsf{F}_r\mathcal{B}(\overline{G}) + \mathsf{F}_{r-1}\mathcal{B}(\overline{G}\setminus\{v\})\\ &\quad + \mathsf{F}_{r-1}\mathcal{B}(\overline{G}) + \mathsf{F}_{r-2}\mathcal{B}(\overline{G}\setminus\{v\})\\ &= \mathsf{F}_{r+1}\mathcal{B}(\overline{G}) + \mathsf{F}_r\mathcal{B}(\overline{G}\setminus\{v\}).\end{aligned}$$ □

Definition 1. Consider r disjoint paths $P_1, \ldots, P_r$ with endpoints u_i, v_i $(i = 1, \ldots, r)$. We call an *eglantine* the graph obtained by adding a vertex c to $\bigcup_{i=1}^r P_i$ and linking c to u_i and v_i $(i = 1, \ldots, r)$. Each subgraph induced by c and one of the paths is called a *petal*, and c is the *center* of the eglantine.

For illustration, all eglantines of order 9 are shown on Figure 2, with the middle vertex as center.

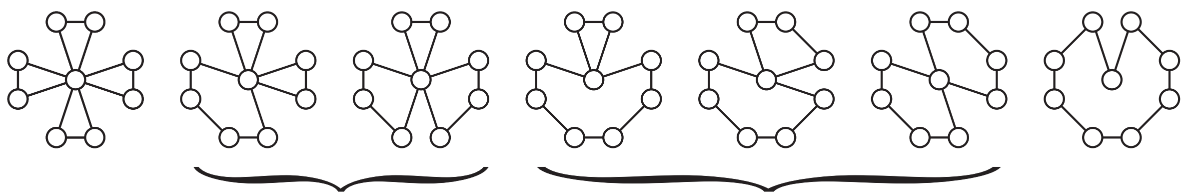

Figure 2: The eglantines of order 9.

Lemma 13. *Let G be the complement of an eglantine of order n with $r \geq 1$ petals. If all petals of $\overline{G}$ have exactly 3 vertices, then*

$$\mathcal{B}(G) = 2^{r-1}(3r + 2).$$

Proof. The proof is by induction on r. If $r = 1$ then $G \simeq \overline{\mathsf{C}_3}$ and $\mathcal{B}(\overline{\mathsf{C}_3}) = 5 = 2^0(3+2)$. So suppose $r \geq 2$ and assume the result is true for smaller values of r. Let c be the center of $\overline{G}$, and let u, v be the two other vertices of a petal in $\overline{G}$. Also, let H denote the eglantine with $r - 1$ petals obtained by removing u and v from $\overline{G}$.

It follows from Equation (2) in Property 1 that $\mathcal{B}(G) = \mathcal{B}(G + cu) + \mathcal{B}(G_{|cu})$. Observe that $G_{|cu}$ is isomorphic to $G \setminus \{c\} \simeq \overline{r\mathsf{K}_2}$ while $\overline{G + cu} = \overline{G} - cu \simeq H_c^2$ (see Figure 3). Hence, we know from Lemma 12 that $\mathcal{B}(G + cu) = 2\mathcal{B}(\overline{H}) + \mathcal{B}(\overline{H}\setminus\{c\})$, where $\overline{H}\setminus\{c\} \simeq \overline{(r-1)\mathsf{K}_2}$. Moreover, by Corollary 5, we have $\mathcal{B}(\overline{r\mathsf{K}_2}) = 2^r$ and $\mathcal{B}(\overline{(r-1)\mathsf{K}_2}) = 2^{r-1}$. In summary, it follows from the induction hypothesis that

$$\mathcal{B}(G) = 2(2^{r-2}(3(r-1)+2)) + 2^{r-1} + 2^r = 2^{r-1}(3r+2).$$

□

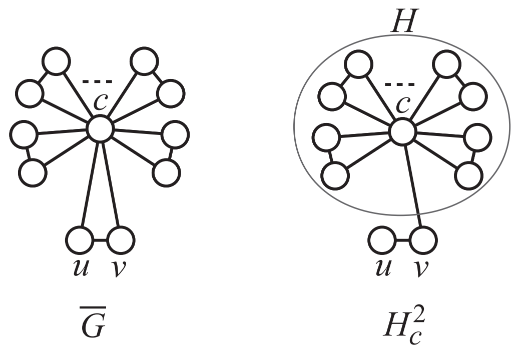

Figure 3: Illustration of the proof of Lemma 13.

We are now ready to prove the main result.

Theorem 14. *Let $G = (V, E)$ be a graph of order $n \geq 3$ and maximum degree $\Delta(G) \leq n-3$. Then*

$$\mathcal{B}(G) \geq h(n)$$

with equality if and only if $G \simeq \mathsf{K}_{n-2} \cup \mathsf{K}_2$ when $n \geq 8$, and $G \simeq \overline{\mathsf{C}_n}$ otherwise.

Proof. We can assume $G \in \mathscr{G}^n_{n-3}$ since $\mathcal{B}(G) > \mathcal{B}(G+uv)$ for every pair of non-adjacent vertices u, v in G. The proof is by induction on n. The theorem is clearly valid for $n = 3$ since $\overline{\mathsf{C}_3}$ is the only graph in $\mathscr{G}^3_0$ and $\mathcal{B}(\overline{\mathsf{C}_3}) = 5 = h(3)$. So assume $n \geq 4$ and suppose G is extremal in $\mathscr{G}^n_{n-3}$. By definition of function $h(n)$, it is sufficient to prove that if G is not isomorphic to $\mathsf{K}_2 \cup \mathsf{K}_{n-2}$ or to $\overline{\mathsf{C}_n}$, then $\mathcal{B}(G) > h(n)$. So assume $G \not\simeq \mathsf{K}_2 \cup \mathsf{K}_{n-2}$ and $G \not\simeq \overline{\mathsf{C}_n}$.

Assume $\overline{G}$ has $k \geq 2$ connected components. Let V_1 be the vertex set of one connected component of $\overline{G}$ and let $V_2 = V \setminus V_1$, with $n_i = |V_i|$. Note that $n_i \geq 3$ because $\Delta(G) = n-3$. Then G is the join of $G[V_1]$ and $G[V_2]$ and it follows from Property 4 that $\mathcal{B}(G) = \mathcal{B}(G[V_1])\mathcal{B}(G[V_2])$. Moreover, $G[V_1] \in \mathscr{G}^{n_1}_{n_1-3}$ and $G[V_2] \in \mathscr{G}^{n_2}_{n_2-3}$. It then follows from the induction hypothesis and Lemma 11 that $\mathcal{B}(G) \geq h(n_1)h(n_2) > h(n)$.

So assume $\overline{G}$ is connected, let A be the set of vertices of degree 2 in $\overline{G}$ and let $B = V \setminus A$. Also, let $W_1, \ldots, W_k$ be the vertex sets of the connected components of $\overline{G}[A]$. Since G is extremal, we know from Property 3 that $\overline{G}[B]$ is a stable set, which means that in $\overline{G}$, every vertex in B has at least three neighbors in A. Moreover, every subgraph $\overline{G}[W_i]$ of $\overline{G}$ is a cycle or a path, and if it is a cycle, it is a connected component of $\overline{G}$. Hence, since $\overline{G}$ is connected and $G \not\simeq \overline{\mathsf{C}_n}$, we have $k \geq 2$ and every $\overline{G}[W_i]$ is a path.

If a set W_i contains only one vertex v, then let x_i and y_i be the two neigbhors of v in $\overline{G}$. If $|W_i| > 1$, then let u_i and v_i be the endpoints of the path $\overline{G}[W_i]$. In $\overline{G}$, u_i has one neigbhor in W_i and another one in B, which we denote by x_i. Also, we denote by y_i the neighbor of v_i in B (with possibly $x_i = y_i$). We now distinguish two cases.

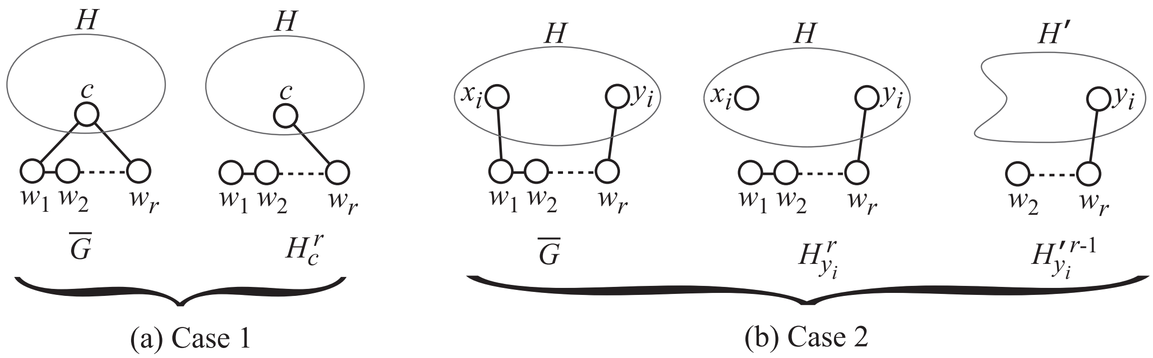

Figure 4: Illustration of the proof of Theorem 14.

Case 1: $x_i = y_i$ for $i = 1, \ldots, k$.

In such a case, every W_i contains at least two vertices, and because $\overline{G}$ is connected, it is an eglantine with $k \geq 2$ petals and a center c equal to x_i and y_i for all $i = 1, \ldots, k$. If all petals have 3 vertices, then $n = 2k + 1$ and we know from Lemma 13 that $\mathcal{B}(G) = 2^{k-1}(3k + 2)$. Since $k \geq 2$ (because we assume $n \geq 4$), it is easy to verify that $2^{k-1}(3k + 2) > 4k^2 - 2k + 1$. Hence

$$\mathcal{B}(G) > 4k^2 - 2k + 1 = (2k + 1)^2 - 3(2k + 1) + 3 = n^2 - 3n + 3 \geq h(n).$$

So assume that at least one petal $\overline{G}[W_i \cup \{c\}]$ $(1 \leq i \leq k)$ has at least 4 vertices. Let $\{c, w_1, \ldots, w_r\}$ $(r \geq 3)$ be the vertex set of this petal and let $\{cw_1, w_1w_2, \ldots, w_{r-1}w_r, w_rc\}$ be its edge set. Also, let H be the eglantine obtained by removing $w_1, \ldots, w_r$ from $\overline{G}$. It follows from Equation (2) in Property 1 that $\mathcal{B}(G) = \mathcal{B}(G + cw_1) + \mathcal{B}(G_{|cw_1})$. Since $\overline{G + cw_1} = \overline{G} - cw_1 \simeq H_c^r$ (see Figure 4 (a)), we know from Lemma 12 that

$$\mathcal{B}(G + cw_1) = \mathsf{F}_{r+1}\mathcal{B}(\overline{H}) + \mathsf{F}_r\mathcal{B}(\overline{H} \setminus \{c\})).$$

Also, $G_{|cw_1}$ is isomorphic to the graph obtained by adding a dominating vertex to $G \setminus \{c, w_1\} \simeq \overline{\mathsf{P}_{r-1}} + \overline{H} \setminus \{c\}$. It then follows from Properties 2, 4 and 6 that

$$\mathcal{B}(G_{|cw_1}) = \mathcal{B}(G \setminus \{c, w_1\}) = \mathsf{F}_r\mathcal{B}(\overline{H} \setminus \{c\}).$$

In summary, we have

$$\mathcal{B}(G) = \mathsf{F}_{r+1}\mathcal{B}(\overline{H}) + 2\mathsf{F}_r\mathcal{B}(\overline{H} \setminus \{c\}).$$

Since $\overline{H}$ has $n - r$ vertices and maximum degree $n - r - 3$, we know by induction that $\mathcal{B}(\overline{H}) \geq h(n - r)$. Also, because $\mathcal{B}(\overline{H} \setminus \{c\})$ has $n - r - 1$ vertices and maximum degree $n - r - 3$, we know from Theorem 8 that $\mathcal{B}(\overline{H} \setminus \{c\}) \geq n - r - 1$. Hence,

$$\mathcal{B}(G)) \geq \mathsf{F}_{r+1}h(n - r) + 2\mathsf{F}_r(n - r - 1)$$

and it follows from Lemma 10 that $\mathcal{B}(G) > h(n)$.

Case 2: there is at least one index i such that $x_i \neq y_i$.

Let $\overline{G}[W_i]$ be a connected component of $\overline{G}[A]$ of maximum order among those for which $x_i \neq y_i$. Suppose W_i contains $r \geq 1$ vertices $w_1, \ldots, w_r$ so that $u_i = w_1$, $v_i = w_r$ and $w_1w_2, \ldots, w_{r-1}w_r$ are the edges in $\overline{G}[W_i]$. Note that $n \geq 5$ because $|B| \geq 2$ and every vertex in B is linked to at least three vertices of A in $\overline{G}$.

It follows from Equation (2) in Property 1 that $\mathcal{B}(G) = \mathcal{B}(G + x_iw_1) + \mathcal{B}(G_{|x_iw_1})$. By denoting $H = \overline{G} \setminus W_i$, we have $\overline{G + x_iw_1} = \overline{G} - x_iw_1 \simeq H^r_{y_i}$ (see Figure 4 (b)), and we know from Lemma 12 that

$$\mathcal{B}(G + x_iw_1) = \mathsf{F}_{r+1}\mathcal{B}(\overline{H}) + \mathsf{F}_r\mathcal{B}(\overline{H} \setminus \{y_i\}).$$

Moreover, because $G_{|x_iw_1}$ is isomorphic to the graph obtained by adding a dominating vertex to $G \setminus \{x_i, w_1\}$, it follows from Property 2 that $\mathcal{B}(G_{|x_iw_1}) = \mathcal{B}(G \setminus \{x_i, w_1\})$. By denoting $H' = H \setminus \{x_i\}$, we have $\overline{G} \setminus \{x_i, w_1\} \simeq {H'}^{r-1}_{y_i}$ (see Figure 4 (b)), and we therefore know from Lemma 12 that

$$\mathcal{B}(G \setminus \{x_i, v_1\}) = \mathsf{F}_r\mathcal{B}(\overline{H'}) + \mathsf{F}_{r-1}\mathcal{B}(\overline{H'} \setminus \{y_i\}).$$

Since $\overline{H}$ has $n - r$ vertices while its maximum degree is $n - r - 3$, we know by induction that $\mathcal{B}(\overline{H}) \geq h(n - r)$. Also, because $\overline{H} \setminus \{y_i\}$ and $\overline{H'}$ have $n - r - 1$ vertices while their maximum degree is $n-r-3$, we know from Theorem 8 that both $\mathcal{B}(\overline{H} \setminus \{y_i\})$ and $\mathcal{B}(\overline{H'})$ are at least equal to $n - r - 1$. Moreover, we clearly have $\mathcal{B}(\overline{H'} \setminus \{y_i\}) \geq 1$. Altogether, we therefore get

$$\begin{aligned}
\mathcal{B}(G) &= \mathcal{B}(G + x_iw_1) + \mathcal{B}(G_{|x_iw_1}) \\
&= \mathsf{F}_{r+1}\mathcal{B}(\overline{H}) + \mathsf{F}_r\mathcal{B}(\overline{H} \setminus \{y_i\}) \\
&\quad + \mathsf{F}_r\mathcal{B}(\overline{H'}) + \mathsf{F}_{r-1}\mathcal{B}(\overline{H'} \setminus \{y_i\}) \\
&\geq \mathsf{F}_{r+1}h(n - r) + 2\mathsf{F}_r(n - r - 1) + \mathsf{F}_{r-1}.
\end{aligned}$$

Since $\mathsf{F}_{r-1} \geq 0$, it follows from Lemma 10 that $\mathcal{B}(G) > h(n)$ if $r \geq 2$. We can therefore assume $r = 1$, which means that $W_i = \{w_1\}$, $\overline{H} \simeq G \setminus \{w_1\}$ and

$$\mathcal{B}(G) \geq \mathcal{B}(G \setminus \{w_1\}) + 2(n - 2) \geq h(n - 1) + 2(n - 2).$$

At this point, it might be helpful to remind some of our assumptions: $G \not\simeq \mathsf{K}_2 \cup \mathsf{K}_{n-2}$ and $G \not\simeq \overline{\mathsf{C}_n}$; $n \geq 5$; $\overline{G}$ is connected; all connected components $\overline{G}[W_1], \ldots, \overline{G}[W_k]$ of $\overline{G}[A]$ are paths; if $x_j \neq y_j$, then W_j contains only one vertex; in particular, $W_i = \{w_1\}$.

- If $n \geq 9$, then $\mathcal{B}(G) \geq (n-1)^2 - 3(n-1) + 3 + 2(n-2) = n^2 - 3n + 3 = h(n)$. In order to have $\mathcal{B}(G) = h(n)$, we must have $\mathcal{B}(G \setminus \{w_1\}) = h(n-1) = (n-1)^2 - 3(n-1) + 3$, which means (by induction) that $G \setminus \{w_1\}$ is isomorphic to $K_2 \cup K_{n-3}$, and G is therefore isomorphic to $K_2 \cup K_{n-2}$, a contradiction. So $\mathcal{B}(G) > h(n)$.

- If $n = 8$, then it is easy to verify that there are only five graphs G that satisfy all our assumptions. Their complements are represented on Figure 5 and their number $\mathcal{B}(G)$ of non-equivalent colorings are respectively equal to 73, 64, 55, 57 and 59. All these numbers are strictly larger than $\mathcal{B}(K_2 \cup K_6) = 43$.

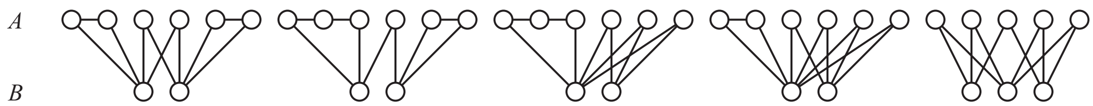

Figure 5: Case $n = 8$.

- If $n = 7$, then it is easy to verify that there are only two graphs G that satisfy all our assumptions. Their complements are shown on Figure 6 and their number $\mathcal{B}(G)$ of non-equivalent colorings are respectively equal to 38 and 45. These two numbers are strictly larger than $\mathcal{B}(\overline{\mathsf{C}_7}) = 29$.

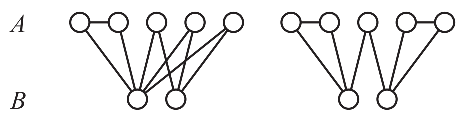

Figure 6: Case $n = 7$.

- If $n = 6$, then $\mathcal{B}(G) \geq h(5) + 2(6-2) = 19 > 18 = h(6)$.

- If $n = 5$, then $\mathcal{B}(G) \geq h(4) + 2(5-2) = 13 > 11 = h(5)$. $\square$

5 Concluding remarks

We have determined a sharp lower bound on the graphical Bell number $\mathcal{B}(G)$ of graphs G of order n and maximum degre $n-3$. We have also characterized the extremal graphs in $\mathscr{G}^n_{n-3}$. Similar results were obtained in [9] for graphs with maximum degree $r = 1, 2, n-2$ and $n-1$. It would be interesting to determine a sharp lower bound on $\mathcal{B}(G)$ for graphs in $\mathscr{G}^n_r$ with r in $\{3, 4, \ldots, n-4\}$. The extremal graphs in this case do not seem to have a simple structure, as was the case for $\Delta(G) = 1, 2, n-3, n-2, n-1$. Indeed, we have determined some of them by exhaustive enumeration, and as an example, we represent in Figure 7 the only graphs G of order $n = 7, 8, 9$ with minimum value $\mathcal{B}(G)$ when $\Delta(G) = n - 4$.

Notice also that several graphs with minimum value $\mathcal{B}(G)$ are non-connected. This is the case, for example for $\mathsf{K}_2 \cup \mathsf{K}_{n-2}$ in $\mathscr{G}^n_{n-3}$, and for $\mathsf{K}_1 \cup \mathsf{K}_{n-1}$ in $\mathscr{G}^n_{n-2}$. It would be interesting to determine such extremal graphs with the additional constraint that G must be connected. Also, it could be interesting to characterize the graphs G that minimize or maximize $\mathcal{B}(G)$ when the order and the size of G are fixed.

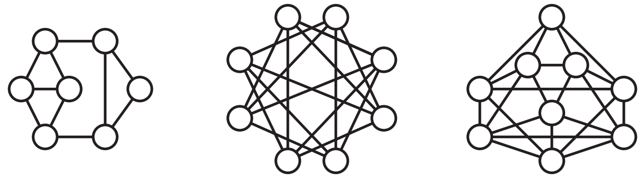

Figure 7: Extremal graphs G with maximum degree $\Delta(G) = n - 4$.

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